**Q1. Explain Total Probability.**

The law of total probability will allow us to use the multiplication rule to find probabilities in more interesting examples. It involves a lot of notation, but the idea is fairly simple. We state the law when the sample space is divided into 3 pieces. It is a simple matter to extend the rule when there are more than 3 pieces.

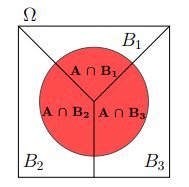
**Law of Total Probability:**

Suppose the sample space Ω is divided into 3 disjoint events B1, B2, B3 (see the figure below). Then for any event A:

P(A) = P(A ∩ B1) + P(A ∩ B2) + P(A ∩ B3) ------------------------(1)

P(A) = P(A|B1) P(B1) + P(A|B2) P(B2) + P(A|B3) P(B3) -------(2)

The top equation (1) says ‘if A is divided into 3 pieces then P(A) is the sum of the probabilities of the pieces’. The bottom equation (2) is called the law of total probability. It is just a rewriting of the top equation using the multiplication rule.



The sample space Ω and the event A are each divided into 3 disjoint pieces. The law holds if we divide Ω into any number of events, so long as they are disjoint and cover all of Ω. Such a division is often called a partition of Ω.

**Q2. Explain Bayes’ theorem in terms of total probability.**

Bayes’ theorem is a pillar of both probability and statistics and it is central to the rest of this course. For two events A and B Bayes’ theorem (also called Bayes’ rule and Bayes’ formula) says



1. Bayes’ rule tells us how to ‘invert’ conditional probabilities, i.e. to find P(B|A) from P(A|B).

2. In practice, P(A) is often computed using the law of total probability.

**Proof of Bayes’ rule:** The key point is that A ∩ B is symmetric in A and B. So the multiplication rule says

P(B|A) · P(A) = P(A ∩ B) = P(A|B) · P(B).

Now divide through by P(A) to get Bayes’ rule.

A common mistake is to confuse P(A|B) and P(B|A). They can be very different. This is illustrated in the next example.

**Example**. Toss a coin 5 times. Let H1 = ‘first toss is heads’ and let HA = ‘all 5 tosses are heads’. Then P(H1|HA) = 1 but P(HA|H1) = 1/16.

For practice, let’s use Bayes’ theorem to compute P(H1|HA) using P(HA|A1). The terms are P(HA|H1) = 1/16, P(H1) = 1/2, P(HA) = 1/32. So,



which agrees with our previous calculation.

**Q3. Oscar has lost his dog in either forest A with a priori probability 0.4 or in forest B with a priori probability 0.6. If the dog is alive and not found by the nth day of search it will die that evening with a probability of n/(n+2). If the dog is in A (either dead or alive) and Oscar spends a day searching for it in A, the conditional probability that he will find it that day is 0.25. Similarly, if the dog is in B and Oscar spends a day searching for it in B with a conditional probability 0.15. The dog can’t go from one forest to the other. Oscar can only search in day time and he can travel from forest A to forest B in night.**

1. **In which forest should Oscar look to maximize the probability that he finds his dog on the first day of search?**
2. **Given that Oscar looked in forest A on the first day but didn’t find his dog, what is the probability that the dog is in forest A?**
3. **Oscar has decided to look in forest A for the first two days, what is the probability that he will find the dog live in second day.**

Let A be the event that the dog was lost in forest A and Ac be the event that the dog was lost in forest B. Let Dn be the event that the dog dies on the nth day. Let Fn be the event that the dog is found on the nth day. Let Sn be the event that Oscar searches forest A on nth day and Snc be the event that he searches forest B on day n.

Given that:

P(A) = 0.4

P(Ac) = 0.6

P(Dn+1 | Dnc, Fnc) = N / (N+2)

P(Fn | A, Sn, Fn-1c) = 0.25

P(Fn | A, Snc, Fcn-1) = 0.15

1. **In which forest should Oscar look to maximize the probability that he finds his dog on the first day of search?**

The probability of finding the dog if he searched in A is given by:

P(F1 | SA1) = P(A)\*P(F1 | SA1 & A) + P(not A)\*P(F1 | SA1 & not A)

= (0.4)(0.25) + (0.6)(0)

= 0.1

The probability of finding the dog if he searched in B is given by:

P(F1 | SB1) = P(B)\*P(F1 | SB1 & B) + P(not B)\*P(F1 | SB1 & not B)

= (0.6)(0.15) + (0.4)(0)

= 0.09

P(F1 | SA1) > P(F1 | SB1). 🡪 Therefore, Oscar should search in forest A.

1. **Given that Oscar looked in forest A on the first day but didn’t find his dog, what is the probability that the dog is in forest A?**

For this part, we assume that A and SA1 are independent events.

P(A | SA1 & F1c)= [P(A & SA1 & F1c)] / [P(SA1 & F1c)]

= [P(F1c | A & SA1)\*P(A & SA1) ] / [P(SA1 & F1c & A) + P(SA1 & F1c & not A)]

= [0.75\*P(A)\*P(SA1)] / [0.75\*P(A)\*P(SA1) + P(F1c | SA1 & A)\*P(SA1 & Ac)]

= [0.75\*0.4\*P(SA1)] / [0.75\*0.4\*P(SA1) + P(SA1)\*P(Ac)]

= (0.75\*0.4) / (0.75\*0.4 + 0.6)

= 0.3 / (0.3 + 0.6)

= 1/3

1. **Oscar has decided to look in forest A for the first two days, what is the probability that he will find the dog live in second day.**

P(find live dog in A day 2) = P(in A) · P(not find in A day 1|in A) ·P(alive day 2) · P(find day 2|in A)

= 0.4 \* 0.75 \* (1 − 1/3) \* 0.25

= 0.05

**References:**

<https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading3.pdf>